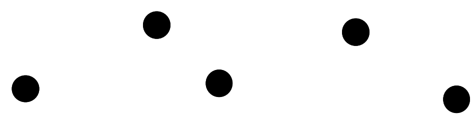
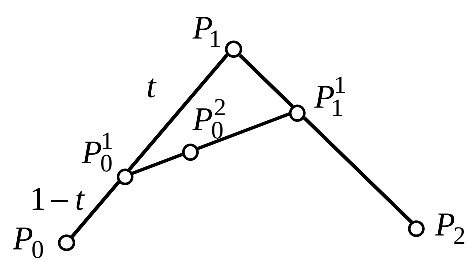
Splines



Functions that interpolate/approximate

1. **Curves and surfaces for modeling**
2. **Keyframes and inbetweens for animation**
3. **Filtering and reconstruction for images**

Chaiken's Algorithm

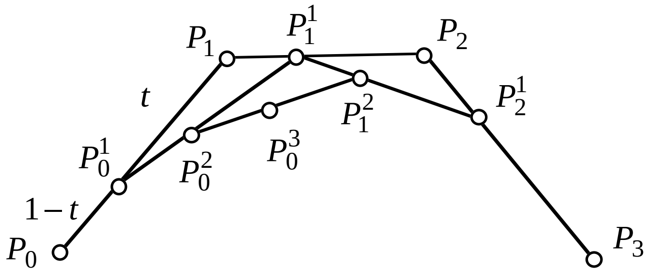


$$P_0^1 = (1-t)P_0 + tP_1$$
$$P_1^1 = (1-t)P_1 + tP_2$$

$$P_0^2 = (1-t)P_0^1 + tP_1^1$$

$$P(t) = P_0^2$$

Bezier Curves



Properties

Property 1: Interpolate end points

Property 2: Tangents

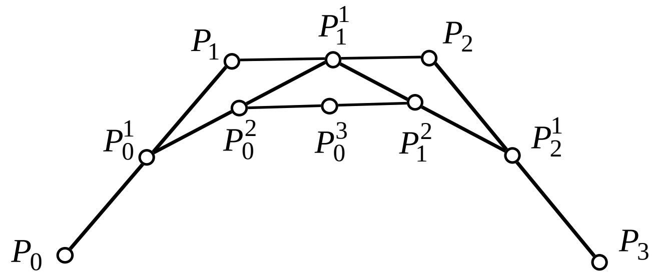
Property 3: Convex hull property

Property 4: Variation diminishing

Property 5: Invariant under affine transformations

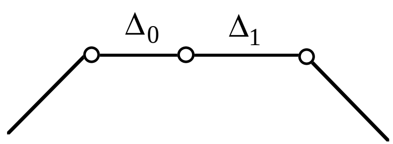
Property 6: Symmetry

Subdivision and Extrapolation Algorithm

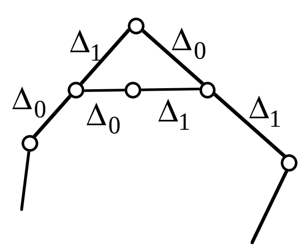


Piecewise Curves

C^1 Continuity



C^2 Continuity



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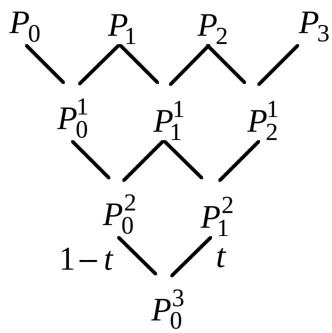
Bernstein Polynomials



$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Matrix Representation of Cubic Curves

$$P(t) = \sum_{i=0}^{3} P_i B_i^n(t) = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix}$$

$$\begin{bmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix}$$

$$\begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}^{-1}$$

Example: Cunic Bezier Curves

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$B_0^3(t) = \binom{3}{0} t^0 (1-t)^3 = -t^3 + 3t^2 - 3t + 1$$

$$B_1^3(t) = \binom{3}{1} t^1 (1-t)^2 = \frac{3!}{2!}\left(t(1-2t+t^2)\right) = 3t^3 - 6t^2 + 3t$$

$$B_2^3(t) = \binom{3}{2} t^2 (1-t)^1 = \frac{3!}{2!}\left(t^2(1-t)\right) = -3t^3 + 3t^2$$

$$B_2^3(t) = \binom{3}{3} t^3 (1-t)^0 = t^3$$

$$\boxed{\begin{bmatrix} B_0^3 \\ B_1^3 \\ B_2^3 \\ B_3^3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix}}$$

Hermite

Given: $P(0), P(1), P'(0), P'(1)$

$$P(t) = at^3 + bt^2 + ct + d$$
$$P'(t) = 3at^2 + 2bt + c$$

$$\begin{bmatrix} H_0 & H_1 & H_2 & H_3 \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$P(0) = H_0 = d$$
$$P(1) = H_1 = a + b + c + d$$
$$P'(0) = H_2 = c$$
$$P'(1) = H_3 = 3a + 2b + c$$

$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix}$$

Hermite Basis Functions

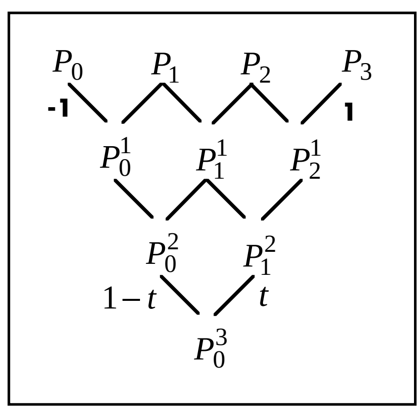
$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix}$$

Finite Differences

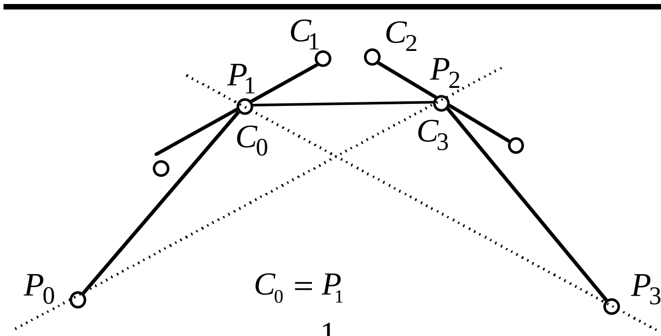
$$\frac{dB_i^n(t)}{dt} = n\left[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right]$$

$$\frac{dP(t)}{dt} = \sum_{i=0}^{n-1} \Delta P_i B_i^{n-1}(t)$$

$$\Delta P_i = P_{i+1} - P_i$$



Catmull-Rom Interpolation



$$C_0 = P_1$$

$$C_1 = \frac{1}{2}(P_2 - P_0)$$

$$C_2 = \frac{1}{2}(P_3 - P_1)$$

$$C_3 = P_2$$

Surfaces ...