Projective Geometry: 2D

Axiom 1
- **2 points determine a line**
- **2 lines determine a point**

Axiom 2
- **2 points incident on 2 lines**
  implies both lines or both points coincident

Axiom 3
- **There exists 4 points, no 3 collinear**
- **There exists 4 lines, no 3 copunctual**

***Duality*: Symmetry between points and lines**

***Parallelism*: 2 lines intersect at point at infinity**

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Homogeneous Coordinates

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \begin{cases} w > 0 & \text{positive finite point} \\ w = 0 & \text{infinite point} \\ w < 0 & \text{negative finite point} \end{cases}$$

- **Multiplying by a non-zero scalar has no effect**
- **For finite points, location equals (x/w,y/w)**
- **For infinite points, direction equal (x,y)**
- **(0,0,0) undefined**

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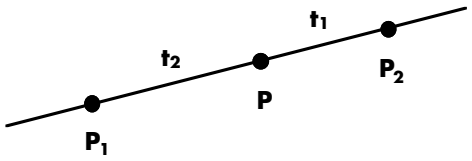
Line From 2 Points



$$P = t_1 P_1 + t_2 P_2$$

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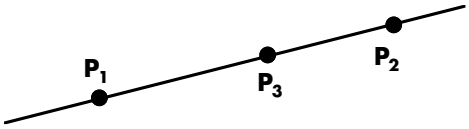
Collinearity



$$t_1 P_1 + t_2 P_2 + t_3 P_3 = 0$$

$$= \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 2\text{area}(\Delta P_1 P_2 P_3)$$

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Line Coordinates

$$\begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ w_1 & w_2 & w \end{vmatrix} = x\begin{vmatrix} y_1 & y_2 \\ w_1 & w_2 \end{vmatrix} + y\begin{vmatrix} w_1 & w_2 \\ x_1 & x_2 \end{vmatrix} + w\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$
$$= ax + by + cw$$

$$a = \begin{vmatrix} y_1 & y_2 \\ w_1 & w_2 \end{vmatrix} = y_1 w_2 - y_2 w_1$$
$$b = \begin{vmatrix} w_1 & w_2 \\ x_1 & x_2 \end{vmatrix} = w_1 x_2 - w_2 x_1$$
$$c = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

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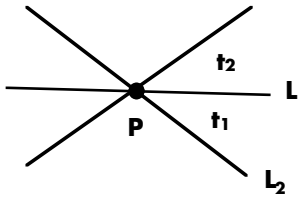
Point From 2 Lines

- ***Pencil* of L's through P**

$$L_1 \bullet P = 0$$
$$L_2 \bullet P = 0$$
$$L \bullet P = (t_1 L_1 + t_2 L_2) \bullet P$$
$$= t_1(L_1 \bullet P) + t_2(L_2 \bullet P)$$
$$= 0$$



$$L = t_1 L_1 + t_2 L_2$$

- **By duality**

$$\vec{L} = \vec{P}_1 \times \vec{P}_2$$
$$\vec{P} = \vec{L}_1 \times \vec{L}_2$$

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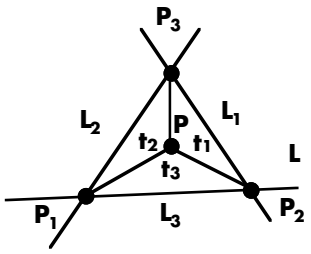
Plane Coordinate Systems

$$P = t_1 P_1 + t_2 P_2 + t_3 P_3$$

$$= \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$



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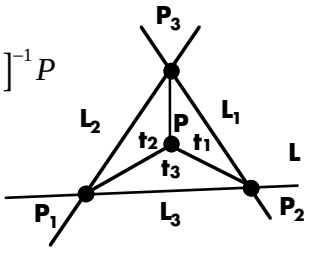
Plane Coordinate Systems

$$P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \Rightarrow \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}^{-1} P$$



$$t_1 = \left| P P_2 P_3 \right| = P \bullet (P_2 \times P_3) = L_1 \bullet P = 2 area(P P_2 P_3)$$

$$t_2 = \left| P_1 P P_3 \right| = P \bullet (P_3 \times P_1) = L_2 \bullet P = 2 area(P_1 P P_3)$$

$$t_3 = \left| P_1 P_2 P \right| = P \bullet (P_1 \times P_2) = L_3 \bullet P = 2 area(P_1 P_2 P)$$

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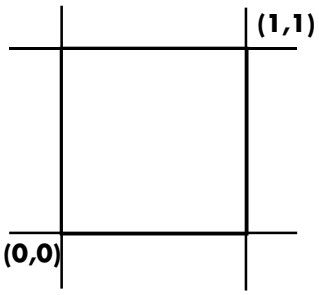
Clipping



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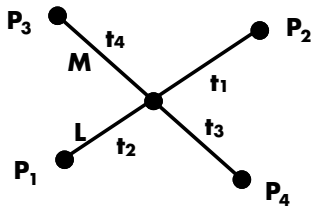
Edge-Edge Intersection



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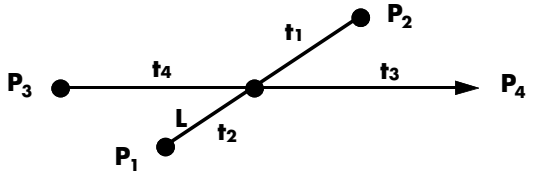
Edge-Ray Intersection



$$t_1 = y_2$$
$$t_2 = -y_1$$
$$t_3 = y_2 - y_1$$
$$t_4 = x_1 y_2 - x_2 y_1$$

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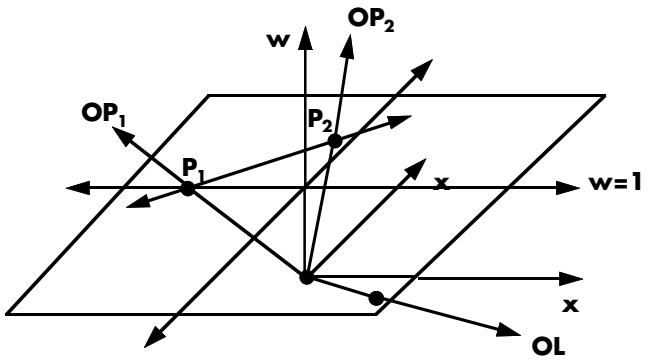
Summary



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