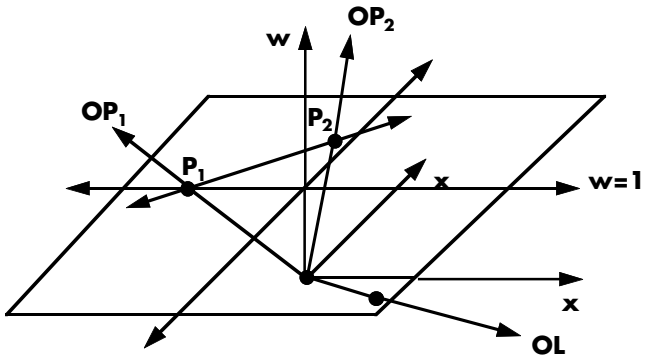
Projective Geometry of Points and Lines



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Homogeneous Coordinates

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad \begin{cases} w > 0 & \text{positive finite point} \\ w = 0 & \text{infinite point} \\ w < 0 & \text{negative finite point} \end{cases}$$

- **Multiplying point by a non-zero scalar has no effect**
- **For finite points, location equals (x/w,y/w,z/w)**
- **For infinite points, direction equal (x,y,z)**
- **(0,0,0,0) undefined**

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Transformations

What?

P′ = T(P)

```
gltranslate(2,2,0);
glrotate(45.,'z');

square();
```

Why?

Modeling

- **Create objects in natural/convenient coordinates**
- **Multiple instances of a prototype shape**
- **Kinematics of linkages/skeletons - robot animation**

Viewing

- **Windows and device independence**
- **Virtual camera: parallel and persective projections**

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Types of Transformations

Continuous (preserves neighborhoods)

One to one, invertible

Classify by invariants or symmetries

- **Isometry (distance preserved)**
  Reflections (interchanges left-handed and right-handed)
  Rigid body motion: Rotations + Translations
- **Similarity (preserves angles)**
  Uniform scale
- **Affine (preserves parallel lines)**
  Non-uniform scales, shears or skews
- **Collineation (lines remain lines)**
  Perspective
- **Non-linear (lines become curves)**
  Twists, bends, warps, morphs, ...

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Linear Transformations = Matrices

$$T(t_1P_1 + t_2P_2) = t_1T(P_1) + t_2T(P_2)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} & t_{xw} \\ t_{yx} & t_{yy} & t_{yz} & t_{yw} \\ t_{zx} & t_{zy} & t_{zz} & t_{zw} \\ t_{wx} & t_{wy} & t_{wz} & t_{ww} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

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Translations

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$

$$T(t_x, t_y, t_z) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Note: translations can be represented as matrices because of w!

What happens if w != 1?
What happens if w == 0?

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Properties of Translations

1. $T(0,0,0) = I$

2. $T(s_x, s_y, s_z)T(t_x, t_y, t_z) = T(s_x + t_x, s_y + t_y, s_z + t_z)$

3. $T(s_x, s_y, s_z)T(t_x, t_y, t_z) = T(t_x, t_y, t_z)T(s_x, s_y, s_z)$

4. $T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$

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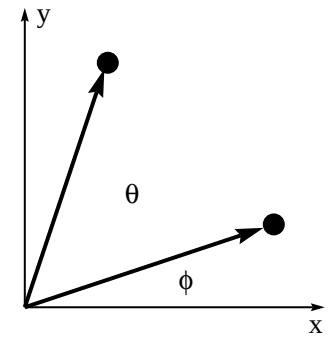
Rotations



$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$x' = r\cos(\phi + \theta)$$
$$y' = r\sin(\phi + \theta)$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi + \theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

$$x' = (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$
$$y' = (r\cos\phi)\sin\theta + (r\sin\phi)\cos\theta$$

$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

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Rotation Matrices

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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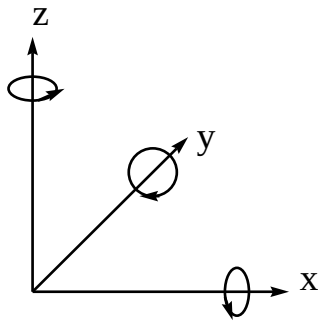
Right-Handed Rule



Right-handed in a right-handed coordinate system
Left-handed in a left-handed coordinate system!

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Properties of Rotations

$$R_a(0) = I$$
$$R_a(\theta)R_a(\phi) = R_a(\theta + \phi)$$
$$R_a(\theta)R_a(\phi) = R_a(\phi)R_a(\theta)$$
$$R_a^{-1}(\theta) = R_a(-\theta) = R_a^T(\theta)$$

But,

$$R_a(\theta)R_b(\phi) \neq R_b(\phi)R_a(\theta)$$ **Order matters**

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Scale

$$x' = ax$$
$$y' = by$$
$$z' = cz$$

$$S(a,b,c) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Other Types of Transformations

1. Rotations about an arbitrary axis

2. Reflections about an arbitrary plane

3. Shear along a plane

4. Perspective and orthographic projections

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