Point set topology is something that every analyst should know something about, but it's easy to get carried away and do too much – it's like candy!
— Ron Getoor (UCSD), 1997 (quoted by Jason Lee)

1 Point Set Topology

In this section, we look at a major branch of topology: point set topology. This branch is devoted to the study of continuity. Developed in the beginning of the last century, point set topology was the culmination of a movement of theorists who wished to place mathematics on a rigorous and unified foundation. The theory is analytical and is therefore not suitable for computational purposes. The concepts, however, are foundational. Therefore, it is important to become familiar with them, as we will see them later, when studying combinatorial topology.

We know that topology is concerned with *connectivity*, and therefore the *neighborhoods* of points. We have actually seen neighborhoods before. In studying high-school calculus, you may have dealt with *epsilon-delta* definition of a limit (or continuity):

Definition 1.1 (Limit) Let $f : D \to \mathbb{R}, D \subseteq \mathbb{R}$ be a function with domain $D \subseteq R$. The limit $\lim_{x \to x_0} f(x) = y_0$ iff for all $\epsilon > 0$, $\exists \delta > 0$ such that if $x \in D$ and $|x - x_0| < \delta$, then $|f(x) - y_0| < \epsilon$.

Note that the definition requires that x falls within an open interval of size δ around x_0. The function, then, maps x to another open interval of size ϵ around the limit value y_0. Open intervals and disks are natural neighborhoods in a Euclidean world. We take their existence for granted because we know how to measure distances (the *bars* in the definition), so we know who is *near* to us. Our ability to measure distances (a metric) gives us the neighborhoods, and therefore our topology.

But suppose we didn't have a metric. We still need neighborhoods to talk about connectivity. Topology formalizes this notion using set theory. If you need to brush up on sets and their operations, read Section 1.4 first.

1.1 Topological Spaces

We begin with a set of X objects we call *points*. Both sets and points are primitive notions, that is, we cannot define them. These points are not in any space yet. We endow our set with structure by using a topology to get a topological space.

Definition 1.2 (topology) A *topology* on a set X is a subset $T \subseteq 2^X$ such that:

1. If $S_1, S_2 \in T$, then $S_1 \cap S_2 \in T$.
2. If $\{S_J \mid j \in J\} \subseteq T$, then $\cup_{j \in J} S_j \in T$.
3. $\emptyset, X \in T$.

The definition states implicitly that only finite intersections, and infinite unions, of the open sets are open. A topology is simply a system of sets that describe the connectivity of the set. These sets have names:

Definition 1.3 (open, closed) Let X be a set and T be a topology. $S \in T$ is an *open set*. The complement of an open set is *closed*.

A set may be only closed, only open, both open and closed, or neither. For instance, $\emptyset$ is both open and closed by definition. These sets are precisely the neighborhoods that we will use to define topology. We combine a set with a topology to get the spaces we are interested in.

Definition 1.4 (topological space) The pair (X, T) of a set X and a topology T is a *topological space*.

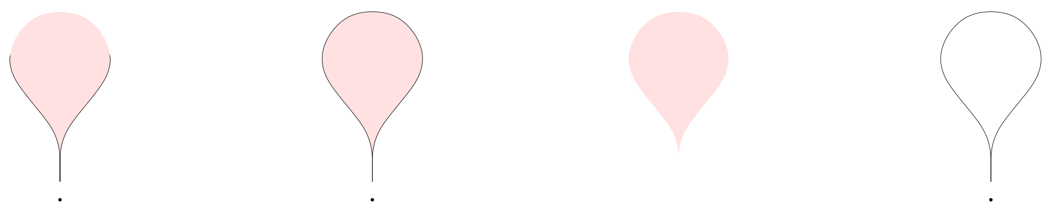
We often use $\mathbb{X}$ as notation for a topological space X, with T being understood.

Definition 1.5 (continuous) A function $f : \mathbb{X} \to \mathbb{Y}$ is continuous if for every open set A in $\mathbb{Y}$, $f^{-1}(A)$ is open in $\mathbb{X}$. We call a continuous function a *map*.

Compare this definition with Definition 1.1. We next turn our attention to the individual sets.

Definition 1.6 (interior, closure, boundary) Let $A \subseteq \mathbb{X}$. The *closure* $\overline{A}$ *of* A is the intersection of all closed sets containing A. The *interior* $\mathring{A}$ *of* A is the union of all open sets contained in A. The *boundary* ∂A *of* A is $\partial A = \overline{A} - \mathring{A}$.

In Figure 1, we see a set that is composed of a single point and a upside-down teardrop shape. We also see its closure,



(a) $A \subseteq \mathbb{X}$ (b) $\overline{A}$ (c) $\mathring{A}$ (d) ∂A

Figure 1. A set $A \subseteq \mathbb{X}$ and related sets.

interior, and boundary. There are other equivalent ways of defining these concepts. For example, we may think of the boundary of a set as the set of points all of whose neighborhoods intersect both the set and its complement. Similarly, the closure of a set is the minimum closed set that contains the set. Using open sets, we can now define neighborhoods.

Definition 1.7 (neighborhoods) Let $\mathbb{X} = (X, T)$ be a topological space. A *neighborhood* of $x \in X$ is any $A \subseteq T$ such that $x \in \mathring{A}$. A *basis of neighborhoods at* $x \in X$ is a collection of neighborhoods of x such that every neighborhood of x contains one of the basis neighborhoods.

Given a topological space $\mathbb{X} = (X, T)$, we may induce topology on any subset $A \subseteq X$. We get the *relative* (or *induced*) topology T_A by defining

$$T_A = \{S \cap A \mid S \in T\}. \tag{1}$$

It is easy to verify that T_A is, indeed, a topology on A, upgrading A to topological space $\mathbb{A}$.

Definition 1.8 (subspace) A subset $A \subseteq X$ with induced topology T_A is a (topological) *subspace* of $\mathbb{X}$.

The important point to keep in mind is that the same set of points may be endowed with different topologies. This is very counter-intuitive at first, but will become clear when we learn about immersions.

1.2 Metric Spaces

As in the definition of limit earlier, we are more familiar with open sets that come from a metric. Let's look at metric spaces next, as they are useful places within which we shall place other spaces.

Definition 1.9 (metric) A *metric* or *distance function* $d : X \times X \to \mathbb{R}$ is a function satisfying the following axioms:

1. For all $x, y \in X$, $d(x, y) \geq 0$ (positivity).
2. If $d(x, y) = 0$, then $x = y$ (non-degeneracy).
3. For all $x, y \in X$, $d(x, y) = d(y, x)$ (symmetry).
4. For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

Definition 1.10 (open ball) The *open ball* $B(x, r)$ with center x and radius $r > 0$ with respect to metric d is defined to be $B(x, r) = \{y \mid d(x, y) < r\}$.

A metric space is a topological space. We can show that open balls can serve as basis neighborhoods for a topology of a set X with a metric.

Definition 1.11 (metric space) A set X with a metric function d is called a *metric space*. We give it the *metric topology* of d, where the set of open balls defined using d serve as basis neighborhoods.

The most familiar of the metric spaces are the Euclidean spaces, where we use the Euclidean metric to measure distances. Below, we use the Cartesian coordinate functions u_i (Definition 1.20 in the appendix.)

Definition 1.12 (Euclidean space) The Cartesian product of n copies of $\mathbb{R}$, the set of real numbers, along with the *Euclidean metric* $d(x,y) = \sqrt{\sum_{i=1}^{n}(u_i(x) - u_i(y))^2}$ is the *n-dimensional Euclidean space* $\mathbb{R}^n$.

We are most familiar with spaces that are subsets of Euclidean spaces. For example, if we have a circle sitting in $\mathbb{R}^2$, we may measure the distance between points on the circle using the metric on $\mathbb{R}^2$. This is the length of the chord connecting the two points. When we do so, we are using the topology induced by $\mathbb{R}^2$ to endow the circle with a topology. We might, however, like to have the distance between the two points on the circle itself. This is a different metric and a different neighborhood basis.

1.3 Manifolds

Manifolds are a type of topological space that we are interested in. In a sense, they are a generalization of Euclidean spaces. Intuitively, a manifold is a topological space that is locally Euclidean. A two-dimensional manifold is *locally flat*: locally, it looks like a plane. If we were living on a space like a sphere, we would think we are living on the plane. In fact, we did.

To define manifolds, we look at maps between topological spaces.

Definition 1.13 (homeomorphism) A *homeomorphism* $f : \mathbb{X} \to \mathbb{Y}$ is a 1-1 onto function, such that both f, f^{-1} are continuous. We say that $\mathbb{X}$ is *homeomorphic* to $\mathbb{Y}$, $\mathbb{X} \approx \mathbb{Y}$, and that $\mathbb{X}$ and $\mathbb{Y}$ have the same *topological type*.

Later, we will use homeomorphisms to define a classification of spaces. For now, we use homeomorphisms to define charts, as shown in Figure 2.

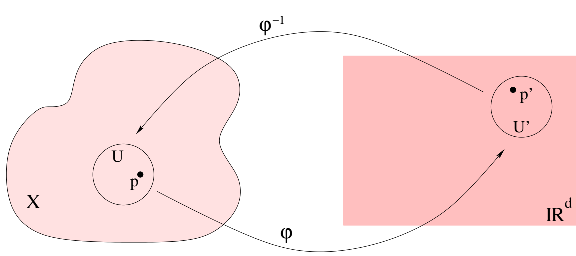


Figure 2. A chart at $p \in \mathbb{X}$. φ maps $U \subset \mathbb{X}$ containing p to $U' \subseteq \mathbb{R}^d$. As φ is a homeomorphism, φ^{-1} also exists and is continuous.

Definition 1.14 (chart) A *chart* at $p \in \mathbb{X}$ is a function $\varphi : U \to \mathbb{R}^d$, where $U \subseteq \mathbb{X}$ is an open set containing p and φ is a homeomorphism onto an open subset of $\mathbb{R}^d$. The *dimension* of the chart φ is d. The *coordinate functions* of the chart are $x^i = u^i \circ \varphi : U \to \mathbb{R}$, where $u^i : \mathbb{R}^n \to \mathbb{R}$ are the standard coordinates on $\mathbb{R}^d$.

We need two additional technical definitions, before we may define manifolds. These definitions rule out really strange spaces which we will never see. I include them so that they do not get endowed with a sense of magic and mystery.

Definition 1.15 (Hausdorff) A topological space $\mathbb{X}$ is *Hausdorff* if for every $x, y \in X, x \neq y$, there are neighborhoods U, V of x, y, respectively, such that $U \cap V = \emptyset$.

The classic example of a non-Hausdorff space is the real line with the origin duplicated as a different point. *All* the neighborhoods of the two origins intersect, but they are different points! A metric space, however, is always Hausdorff.

Definition 1.16 (separable) A topological space $\mathbb{X}$ is *separable* if it has a countable basis of neighborhoods.

Countable means having the same cardinality as integers, that is, the infinity all of us are familiar with (there are bigger ones, such as the cardinality of real numbers.) Again, metric spaces are separable (it's relatively easy to see this in Euclidean space, as an irrational point is always near a rational one.) Finally, we can formally define a manifold.

Definition 1.17 (manifold) A separable Hausdorff space $\mathbb{X}$ is called a *(topological, abstract) d-manifold* if there is a d-dimensional chart at every point $x \in \mathbb{X}$, that is, if $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$. It is called a *d-manifold with boundary* if $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or the Euclidean half-space $\mathbb{H}^d = \{x \in \mathbb{R}^d \mid x_1 \geq 0\}$. The *boundary $\partial\mathbb{X}$ of $\mathbb{X}$* is the set of points with neighborhood homeomorphic to $\mathbb{H}^d$. The manifold has *dimension d*.

Figure 3 displays a 2-manifold, and a 2-manifold with boundary.

Figure 3. The sphere (left) is a 2-manifold. The torus with two holes (right) is a 2-manifold with boundary. Its boundary, a 1-manifold, is composed of the two circles.

Theorem 1.1 *The boundary of a d-manifold with boundary is a $(d-1)$-manifold without boundary.*

The manifolds shown are compact.

Definition 1.18 (compact) A *covering of $A \subseteq X$* is a family $\{C_j \mid j \in J\}$ in 2^X, such that $A \subseteq \bigcup_{j \in J} C_j$. An *open covering* is a covering consisting of open sets. A *subcovering* of a covering $\{C_j \mid j \in J\}$ is a covering $\{C_k \mid k \in K\}$, where $K \subseteq J$. $\mathbb{A} \subseteq \mathbb{X}$ is *compact* if every open covering of A has a finite subcovering.

Intuitively, you might think any finite area manifold is compact. However, a manifold can have finite area and not be compact, such as the cusp in Figure 4.

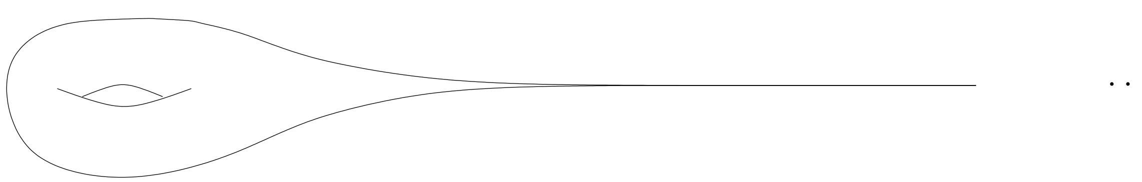


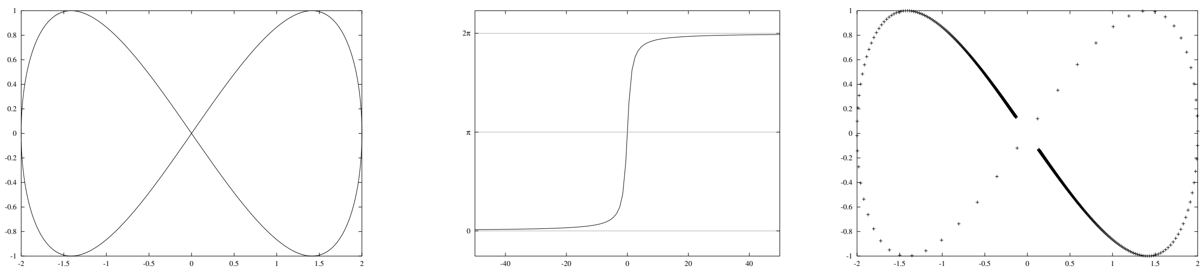
Figure 4. The cusp has finite area, but is not compact

A homeomorphism allows us to place one manifold within another.

Definition 1.19 (embedding) An *embedding* $g : \mathbb{X} \to \mathbb{Y}$ is a homeomorphism onto its image $g(\mathbb{X})$. The image is called an *embedded submanifold* and it is given its relative topology in $\mathbb{Y}$.

Most of our interaction with manifolds in our lives has been with embedded manifolds in Euclidean spaces. Consequently, we always think of manifolds in terms of an embedding. It is important to remember that a manifold exists independently of any embedding: a sphere does not have to sit within $\mathbb{R}^3$ to be a sphere. This is, by far, the biggest shift in the view of the world required by topology.

Example 1.1 Figure 1.1 (a) shows an map of $\mathbb{R}$ into $\mathbb{R}^2$. Note that while the map is 1-1 locally, it is not 1-1 globally. The map F wraps $\mathbb{R}$ over the figure-eight over and over. Using the monotone function g in Figure 1.1 (b), we first fit all of $\mathbb{R}$ into the interval $(0, 2\pi)$ and then map it using F once again. We get the same image (figure-eight) but cover it only once, making $\hat{F}$ 1-1. However, the graph of $\hat{F}$ approaches the origin in the limit, at both ∞ and $-\infty$. Any neighborhood of the origin within $\mathbb{R}^2$ will have four pieces of the graph within it and will not be homeomorphic to $\mathbb{R}$. Therefore, the map is not homeomorphic to its image and not an embedding.



(a) $F(t) = 2\cos(t - \pi/2), \sin(2(t - \pi/2))$

(b) $g(t) = \pi + 2\tan^{-1}(t)$

(c) $\hat{F}(t) = F(g(t))$

Figure 5. Mapping of $\mathbb{R}$ into $\mathbb{R}^2$ with topological consequences.

The maps shown in Figure 1.1 are both immersions. Immersions are defined for smooth manifolds, which are described in further detail in the second appendix (for those of you who think differential manifolds are like candy.) If our original manifold $\mathbb{X}$ is compact, nothing "nasty" can happen. an *immersion* $F : \mathbb{X} \to \mathbb{Y}$ is simply a *local* embedding. In other words, for any point $p \in \mathbb{X}$, there exists a neighborhood U containing p such that $F|_U$ is an embedding. However, F need not be an embedding within the neighborhood of $F(p)$ in $\mathbb{Y}$. That is, immersed compact spaces may self-intersect.

Acknowledgments

Most of the material of this section is from Bishop and Goldberg [1] and Boothby [2]. I also used Henle [3] and McCarthy [4] for reference and inspiration. I would like to thank Ileana Streinu for her close reading of the notes, her many questions, and the lively discussion that clarified the lecture.

References

[1] Bishop, R. L., and Goldberg, S. I. *Tensor Analysis on Manifolds*. Dover Publications, Inc., New York, 1980.

[2] Boothby, W. M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*, second ed. Academic Press, San Diego, CA, 1986.

[3] Henle, M. *A Combinatorial Introduction to Topology*. Dover Publications, Inc., New York, 1997.

[4] McCarthy, G. *Topology: An Introduction with Application to Topological Groups*. Dover Publications, Inc., New York, 1988.

$R = \{x : x \notin x\}$. Then, $R \in R$ iff $R \notin R$.
— Bertrand Russell (1872–1970)

1.4 Sets and Functions (Appendix)

We cannot define a set formally, other than stating that a *set* is a well-defined collection of objects. We also assume the following:

1. Set S is made up of *elements* $a \in S$.
2. There is only one empty set $\emptyset$.
3. We may describe a set by characterizing it ($\{x \mid \mathrm{P}(x)\}$), or by enumerating elements ($\{1, 2, 3\}$). Here P is a predicate.
4. A set S is *well-defined* if for each object a, either $a \in S$ or $a \notin S$.

Note that "well-defined" really refers to the definition of a set, rather than the set itself. $|S|$ or $\operatorname{card} S$ is the size of the set. We may multiply sets in order to get larger sets.

Definition 1.20 (Cartesian) *Cartesian product of sets* $S_1, S_2, \ldots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in S_i$. The Cartesian product is denoted by either $S_1 \times S_2 \times \ldots \times S_n$ or by $\prod_{i=1}^{n} S_i$. The *i-th Cartesian coordinate function* $u_i : \prod_{i=1}^{n} S_i \to S_i$ is defined by $u_i(a_1, a_2, \ldots, a_n) = a_i$.

Having described sets, we define subsets.

Definition 1.21 (Subsets) A set B is a *subset* of a set A, denoted $B \subseteq A$ or $A \supseteq B$, if every element of B is in A. $B \subset A$ or $A \supset B$ is generally used for $B \subseteq A$ and $B \neq A$. If A is any set, then A is the *improper subset* of A. Any other subset is *proper*. If A is a set, we denote by 2^A, the *power set of* A, the collection of all subsets of A, $2^A = \{B \mid B \subseteq A\}$.

We also have a couple of fundamental set operations.

Definition 1.22 (Intersection, Union) The *intersection* $A \cap B$ of sets A and B is the set consisting of those elements which belong to both A and B, that is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The *union* $A \cup B$ of sets A and B is the set consisting of those elements which belong to A or B, that is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

We indicate a collection of sets by labeling them with subscripts from an index set J, e.g. A_j with $j \in J$. For example, we use $\bigcap_{j \in J} A_j = \bigcap\{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for all } j \in J\}$ for general intersection. The next definition summarizes functions, maps relating sets to sets.

Definition 1.23 (Relations and Functions) A *relation* φ between sets A and B is a collection of ordered pairs (a, b) such that $a \in A$ and $b \in B$. If $(a, b) \in \varphi$, we often denote the relationship by $a \sim b$. A *function* or *mapping* φ *from a set* A *into a set* B is a rule that assigns to each element a of A exactly one element b of B. We say that φ *maps* a *into* b, and that φ *maps* A *into* B. We denote this by $\varphi(a) = b$. The element b is the *image of* a *under* φ. We show the map as $\varphi : A \to B$. The set A is the *domain of* φ, the set B is the *codomain of* φ, and the set $\operatorname{im} \varphi = \varphi(A) = \{\varphi(a) \mid a \in A\}$ is the *image of* A *under* φ. If φ and ψ are functions with $\varphi : A \to B$ and $\psi : B \to C$, then there is a natural function mapping A into C, the *composite function*, consisting of φ followed by ψ. We write $\psi(\varphi(a)) = c$ and denote the composite function by $\psi \circ \varphi$. A function from a set A into a set B is *one to one (1-1)* (*injective*) if each element B has at most one element mapped into it, and it is *onto* B (*surjective*) if each element of B has at least one element of A mapped into it. If it is both, it's a *bijection*. A bijection of a set onto itself is called a *permutation*.

A permutation of a finite set is usually specified by its action on the elements of the set. For example, we may denote a permutation of the set $\{1, 2, 3, 4, 5, 6\}$ by $(6, 5, 2, 4, 3, 1)$, where the notation states that the permutation maps 1 to 6, 2 to 5, 3 to 2, and so on. We may then obtain a new permutation by a transposition: switching the order of two neighboring elements. In our example, $(5, 6, 2, 4, 3, 1)$ is a permutation that is one transposition away from $(6, 5, 2, 4, 3, 1)$. We may place all permutations of a finite set in two sets.

Theorem 1.2 (Parity) *A permutation of a finite set can be expressed as either an even or an odd number of transpositions, but not both. In the former case, the permutation is* even*. In the latter, it is* odd.

1.5 Smooth Manifolds (Appendix)

We will next look at smooth manifolds. We know what smooth means within the Euclidean domain. It's easy to extend the notion of smoothness to manifolds because we know that they are locally flat; that is, there is a local chart that maps the neighborhood of a point to the Euclidean space.

Definition 1.24 (C^∞) Let $U, V \subseteq \mathbb{R}^d$ be open. A function $f : U \to \mathbb{R}$ is *smooth* or C^∞ (*continuous of order* ∞) if f has partial derivatives of all orders and types. A function $\varphi : U \to \mathbb{R}^e$ is a C^∞ *map* if all its components $e^i \circ \varphi : U \to \mathbb{R}$ are smooth Two charts $\varphi : U \to \mathbb{R}^d, \psi : V \to \mathbb{R}^e$ are C^∞*-related* if $d = e$ and either $U \cap V = \emptyset$ or $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are C^∞ maps. A C^∞ *atlas* is one for which every pair of charts is C^∞-related. A chart is *admissible* to a C^∞ atlas if it is C^∞-related to every chart in the atlas.

C^∞-related charts allow us to pass from one coordinate system to another smoothly in the overlapping region, so we may extend our notions of curves, functions, and differentials easily to manifolds.

Definition 1.25 (C^∞ Manifold) A *smooth (C^∞) manifold* is a topological manifold together with all the admissible charts of some C^∞ atlas.

The map used between smooth manifolds is called a diffeomorphism.

Definition 1.26 (diffeomorphism) A *diffeomorphism* $g : \mathbb{X} \to \mathbb{Y}$ is a C^∞ map that is a homeomorphism and whose inverse g^{-1} is C^∞. We say that $\mathbb{X}$ is *diffeomorphic* to $\mathbb{Y}$.

A diffeomorphism g allows us to place a smooth manifold $\mathbb{X}$ within another smooth manifold $\mathbb{Y}$. We would like to know more about the image $g(\mathbb{X}) \subseteq \mathbb{Y}$. To do so, we take advantage of the atlas on each manifold. Suppose that U, φ is a chart at $p \in \mathbb{X}$ and V, ψ is a chart at $g(p) \in \mathbb{Y}$. This allows us to get an expression for g in terms of local coordinates:

$$\begin{array}{ccc} U \subseteq \mathbb{X} & \xrightarrow{g} & V \subseteq \mathbb{Y} \\ \varphi \big\downarrow & & \psi \big\downarrow \\ \varphi(U) \subseteq \mathbb{R}^d & \xrightarrow{\hat{g}} & \psi(V) \subseteq \mathbb{R}^e \end{array}$$

That is, $\hat{g} = \psi \circ g \circ \varphi^{-1}$.

Definition 1.27 (Jacobian) The *Jacobian matrix* Dg of a map $g : \mathbb{X} \to \mathbb{Y}$ with local charts U, φ at $p \in \mathbb{X}$ and V, ψ is a chart at $g(p) \in \mathbb{Y}$ is:

$$\frac{\partial(g^1, \ldots, g^e)}{\partial(x^1, \ldots, x^d)} = \begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \cdots & \frac{\partial g^1}{\partial x^d} \\ \vdots & & \vdots \\ \frac{\partial g^e}{\partial x^1} & \cdots & \frac{\partial g^e}{\partial x^d} \end{pmatrix}$$

Dg is defined at each point of U, its $d \cdot e$ entries being functions on U.

The rank of the Jacobian tells us what the diffeomorphism does to its domain space.

Definition 1.28 (rank) The rank of g is the rank of Dg.

This rank is independent of the coordinate system we use (and can be defined independently, too, but that's beyond the scope of this class.)

Definition 1.29 (immersion) $g : \mathbb{X} \to \mathbb{Y}$ is an *immersion* if $\operatorname{rank} g = \dim \mathbb{X}$.

Intuitively, An immersion places a space within another one so that its dimension does not change, and it doesn't develop any kinks. The immersed space, however, can intersect itself or behave in otherwise unappetizing ways, as we saw in Example 1.1. What we are really after are nice immersions, or embeddings.

Definition 1.30 (embedding) An *embedding* $g : \mathbb{X} \to \mathbb{Y}$ is a 1-1 immersion that is a homeomorphism onto its image $g(\mathbb{X})$ considered as a subspace of $\mathbb{Y}$. The image is called an *embedded submanifold* and is given the relative topology.

The definition of smooth manifolds also allows us to give a point-set theoretic definition of orientability. We will see later that the following definitions also apply in non-smooth spaces, such as simplicial spaces.

Definition 1.31 (orientability) A pair of charts x^i and y^i is *consistently oriented* if the Jacobian determinant $\det(\partial x^i / \partial y^j)$ is positive whenever defined. A manifold M is *orientable* if there exists an atlas such that every pair of coordinate systems in the atlas is consistently oriented. Such an atlas is *consistently oriented* and determines an *orientation on* M. If a manifold is not orientable, it is *unorientable*.

In other words, a manifold of any dimension falls into two classes, depending on whether it is orientable or not.